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A New Physical Characterisation of Classical Systems in Quantum Mechanics

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Abstract

A physical characterisation of classical systems in quantum mechanics is given in terms of the set of ensembles in contrast to the well-known characterisations concerning the effects or observables: A quantum mechanical system is classical if and only if each two decompositions of every ensemble are compatible.

1. *Introduction*

The investigation of classical systems in the frame of axiomatic quantum mechanics is of considerable relevance for several reasons.

A physical characterisation of classical systems always gives some information about what purely quantum mechanical systems are not.

On the other hand the connection between observables and the description of classical systems allows us to carry over results valid for classical systems to regions of quantum mechanical systems (Neumann, 1971).

Throughout this paper it is assumed that a quantum mechanical system is described by a separable base normed Banach space B and its dual B' (Davies & Lewis, 1970; Ludwig, 1972). B' is an order unit space. The base K of the closed positive cone of B represents the set of ensembles and the order interval $L = \{f \in B'/0 \leqslant f \leqslant 1\}$ represents the set of effects (yes-no experiments), 1 being the order unit in B'. The probability of measuring an effect $f \in L$ in the ensemble $v \in K$ is given by the value of the functional f on v or in an equivalent formulation by the value of the canonical bilinear form μ on $B \times B'$.

In addition to this assumption we suppose that the axioms 4a, 4bz, 5 of Ludwig (1972) are satisfied though no explicit use is made of them in Section 2. Axiom 4a implies that for all $v \in K$ the sets $\{v\}^\perp \cap L = \{f \in L/\mu(v, f) = 0\}$ contain greatest, i.e. most sensitive effects which are called decision effects. The set G of decision effects is a complete orthocomplemented orthomodular lattice with respect to the ordering on G induced by the ordering of B'.

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A quantum mechanical system is called classical if the set G of decision effects is coexistent (Neumann, 1971). This is exactly the case if the lattice G is Boolean. A quantum mechanical system is classical if and only if B is a separable L-space and thus B is norm and order isomorphic to a space $L_1(S, \Sigma, m)$ where (S, Σ, m) is a finite measure space. G is σ-isomorphic to Σ/J_m, J_m denoting the ideal of sets of m-measure zero.

2. *The Theorem*

In order to motivate the definition of compatibility of decompositions of ensembles consider an ensemble $v \in K$ of a quantum mechanical system. In general there are many preparing procedures which can be described by the ensemble v. A definite preparing procedure p consists of a set of preparing parts which satisfy the macroscopic criteria given by the preparing procedure p and with each of which a single experiment is performed.† (A repetition of an experiment defines a new preparing part.)

It can occur that there are macroscopic criteria which allow the division of the set p of preparing parts into classes $a_i \subset p, i = 1, \ldots, n$, which are again preparing procedures. In this case the ensemble v describing p is a mixture $v = \Sigma_{i=1}^{n} \lambda_{a_i} v_{a_i}$ ($\lambda_{a_i} \geqslant 0, \Sigma_{i=1}^{n} = 1$) of ensembles v_{a_i} describing the procedures a_i. It is reasonable to assume that the class of subprocedures of a given preparing procedure p is a Boolean algebra A of subsets of p and according to the physical interpretation of a mixture the mapping $\chi: A \to \bigcup_{0<\lambda<1} \lambda K$ defined by $a \to \lambda_a v_a$ is a vector valued measure on A. (If $a, b \in A$ with $a \cap b = \emptyset$ then $\lambda_{a \cup b} v_{a \cup b} = \lambda_a v_a + \lambda_b v_b$).

Two decompositions

$$v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n'} \lambda_i' v_i'$$

of an ensemble v shall be called compatible if one can consider both decompositions generated by a division into classes of a common preparing procedure p of v.

Definition. Two decompositions

$$v = \sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n'} \lambda_i' v_i'$$

with $v, v_i, v_i' \in K, \lambda_i, \lambda_i' \geqslant 0$ are called compatible if there is a Boolean algebra A and a mapping $\chi: A \to \bigcup_{0<\lambda<1} \lambda K$ such that χ is a vector-valued measure on A, $\chi(1) = v$ and

$$\{\lambda_i v_i, \lambda_k' v_k' / i = 1, \ldots, n, k = 1, \ldots, n'\} \subset \chi(A)$$

The notion of compatible decompositions of ensembles is of great importance

† Concerning the notions of preparing parts and preparing procedures in the axiomatic foundation of quantum mechanics, see Ludwig (1974) and Hartkämper (1974).

in the investigation of the preparing and measuring process. The incompatibility of decompositions of ensembles reflects the difficulty of assigning 'real' propositions to single quantum mechanical particles. Incompatible decompositions of an ensemble are considered for instance in the Einstein-Rosen-Podolski paradox. A simple example of incompatible decompositions are the decompositions $W = \frac{1}{2}P_x^+ + \frac{1}{2}P_x^- = \frac{1}{2}P_y^+ + \frac{1}{2}P_y^-$ in the two-dimensional spin-Hilbert space, P_x^+ being the projection operator onto the $+\frac{1}{2}$-spin state in the x-direction and P_x^-, P_y^+, P_y^- being analogously defined.

On the other hand $W = \frac{1}{4}P_x^+ + \frac{1}{4}P_x^- + \frac{1}{4}1 = \frac{1}{4}P_y^+ + \frac{1}{4}P_y^- + \frac{1}{4}1$ are compatible decompositions of this ensemble.

Theorem. A quantum mechanical system described by B, B' is classical if and only if each two decompositions of every ensemble are compatible.

Proof. If each two decompositions of every $v \in K$ are compatible B has the Riesz decomposition property. In order to prove this consider $x_1, x_2 \in B$ with $x_1, x_2 \geqslant 0$. If $[0, x_i] = \{x \in B/0 \leqslant x \leqslant x_i\}$ we have to verify $[0, x_1] + [0, x_2] = [0, x_1 + x_2]$. Consider $z \in [0, x_1 + x_2], z \neq 0$. We may assume $\|x_1 + x_2\| = 1$, hence $v = x_1 + x_2 \in K$. $v = x_1 + x_2$ and $v = (v - z) + z$ are compatible decompositions of v. There is a Boolean algebra A and a vector-valued measure χ: $A \to \bigcup_{0 \leqslant \lambda \leqslant 1} \lambda K$ such that $\chi(1) = v$, $\chi(q_i) = x_i$, $i = 1, 2$, and $\chi(q_3) = v - z$, $\chi(q_4) = z$. Let $\cup$, $\cap$ and $*$ denote the Boolean operations union, intersection and complement in A. Since $q_4 = (q_4 \cap q_1) \cup (q_4 \cap q_1^*)$ we have

$$z = \chi(q_4 \cap q_1) + \chi(q_4 \cap q_1^*)$$

$$\chi(q_4 \cap q_1) \leqslant \chi(q_1) = x_1$$

and

$$\chi(q_4 \cap q_1^*) \leqslant \chi(q_1^*) = v - x_1 = x_2$$

Thus the Riesz decomposition property holds. It is well known that a base normed Banach space with a closed cone having the Riesz decomposition property is an L-space (compare, e.g., Ellis, 1964). B' is a vector lattice and the lattice G of decision effects is a sublattice and hence Boolean.

To prove the inverse implication assume B, B' to describe a classical system. It is sufficient to show that for all $v \in K$ there is a Boolean algebra A and a vector-valued measure χ: $A_v \to \bigcup_{0 \leqslant \lambda \leqslant 1} \lambda K$ such that $\chi(1) = v$ and $M_v = \{x \in B/0 \leqslant x \leqslant v\} \subset \chi(A_v)$.

First of all we shall introduce a product on M_v. According to the statement quoted in the introduction we may assume $B = L_1(S, \Sigma, m)$ for a finite measure space (S, Σ, m). If $x \in B$ let m_x denote the finite measure on S associated with x. For all $x \in M_v$ there is an m_v-integrable function f_x on S such that $m_x(\sigma) = \int_\sigma f_x(s)\, dm_v(s)$ for all $\sigma \in \Sigma$. Moreover $0 \leqslant f_x(s) \leqslant 1$ a.e. on S. If $x_1, x_2 \in M_v$ we have $0 \leqslant f_{x_1}(s) . f_{x_2}(s) \leqslant 1$ and $f_{x_1} . f_{x_2}$ is an m_v-integrable function. Thus there is $x_3 \in M_v$ such that $f_{x_3} = f_{x_2} . f_{x_1}$ and we have introduced a product on M_v. The relations $x_1 . x_2 = x_2 . x_1, x_1 . v = x_1$ hold for all $x_1, x_2 \in M_v$. Moreover $x(x_1 + x_2) = x . x_1 + x . x_2$ if $x, x_1, x_2,$

$x_1 + x_2 \in M_v$. Notice, however, that the definition of the product strongly depends on the fixed $v \in K$.

Let R_{M_v} be the free Boolean algebra generated by the set M_v. If T denotes the set consisting of two elements, $T = \{0, 1\}$, R_{M_v} can be considered as the Boolean algebra of subsets of T^{M_v} generated by the sets $p_x^{-1}(1)$, where T^{M_v} is the set of all mappings $M_v \to T$ and p_x is the canonical projection p_x: $T^{M_v} \to T$ for $x \in M_v$. Then every element of R_{M_v} is a disjoint union of 'monomials'

$a = \bigcap_{i=1}^{\pi} p_{x_i}^{-1}(\delta_i)$, $\delta_i \in T$. A vector-valued measure is defined on R_{M_v} if it is defined on these monomials. We put $A_v = R_{M_v}$ and define

$$\chi(a) = \prod_{i=1}^{n} \delta_i(x_i), \qquad \delta_i = \begin{cases} x_i & \text{if } \delta_i = 1 \\ v - x_i & \text{if } \delta_i = 0 \end{cases}$$

where Π denotes the product introduced in M_v.

Since $\chi(a) = \chi(a) . (v - x) + \chi(a) . x$ for $a \in R_{M_v}, x \in M_v$, χ is a vector-valued measure on R_{M_v} such that $\chi(R_{M_v}) = M_v$ (compare also the proof of Theorem 8 (Neumann, 1971)). This completes the proof of the theorem.

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